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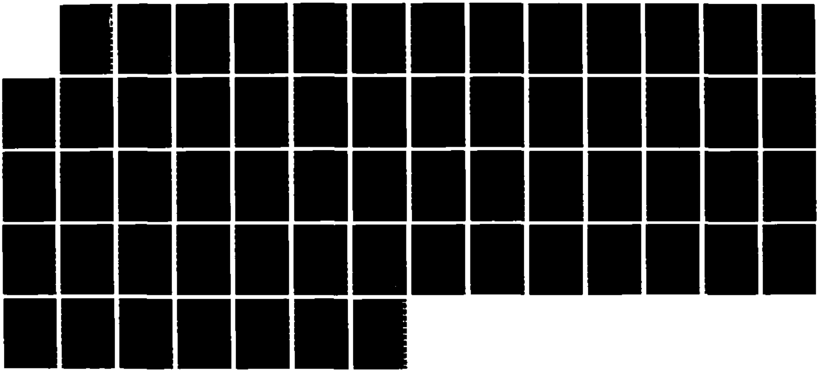

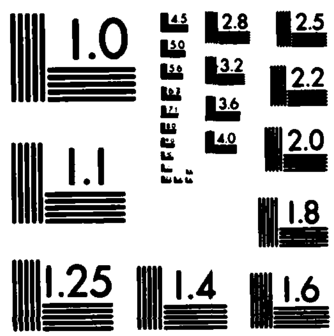
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INTERNAL REPORT

RELIABILITY AS A FUNCTION OF FATIGUE, COMPLEXITY AND REDUNDANCY

A MATHEMATICAL ANALYSIS

OCTOBER 1985

DR BRUNO J. MANZ

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DIRECTORATE OF AEROSPACE STUDIES
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This is a mathematical analysis of the reliability concept and two kinds of reliability functions: The exponential and the Weibull distribution. Both functions are analyzed with respect to the following four determinants of reliability:

1. System complexity,
2. Subsystem criticality,
3. Subsystem redundancy,
4. Fatigue.

Appropriate formulas describing these agencies are developed, and their implications are discussed.

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1. INTRODUCTION

The reliability of a system is a function of many factors. The most important factors which we call "determinants" are listed below.

Determinants

a. System Complexity

b. Subsystem Criticality

c. Subsystem Redundancy

d. Proclivity to Fatigue

A brief discussion follows.

a. Complexity

Given that the subsystems of systems A and B have equal complexity, system A is more complex than System B if A has more subsystems than B. In this case complexity can be measured directly in terms of the number of subsystems N to which we refer as the "complexity index".

b. Criticality

Every subsystem has a certain criticality with regard to either the system or the mission. Subsystem A has higher criticality than subsystem B if failure of A has a stronger negative effect on the system or the mission than failure of B.

c. <u>Redundancy</u>

A given subsystem has redundancy r if it has r spares, that is, if there are altogether $r + 1$ subsystems of the given kind. If $r = 0$, there is no spare and, consequently, there is no redundancy. If $r = 1$, there is one spare, that is, two subsystems of one kind. The redundancy index r can assume nonnegative integer values and is, at least in principle, unlimited on the upside.

d. <u>Fatigue</u>

This is the most difficult concept to explain without the aid of mathematics. Another word for "fatigue" is "aging". We believe that fatigue, or aging, is the only cause of failure; in other words, we cannot think of another cause. But we shall show that there are two distinct mathematical ways to describe the phenomenon of failure, leading to two categories of systems which we call systems of Type 1 and Type 2. The corresponding concepts most frequestly used in the literature are "exponential distribution" and "Weibull distribution". In Section 3, we analyze the implications of both system types or distributions.

The primary subjects of the present report are complexity and redundancy. However, when dealing with subjects of reliability, one can hardly ignore fatigue. This is the reason why we precede the sections on complexity and redundancy with a section on fatigue.

Criticality was addressed in considerable detail in Reference 1. There will also be some cryptic remarks on criticality in Section 4. However, most of the material offered in Reference 1 will not be repeated in the present report, except for some mathematical techniques which will facilitate the exposition of some ideas presented later on.

Ref 1. Bruno J. Manz, Topics from the Theory of Reliability, DAS-WP-77-5, July 1977, Unclassified.

2. RECAPITULATION OF SOME BASIC FORMULAS FROM THE THEORY OF RELIABILITY

In the present section, we limit the consideration to a single subsystem. Therefore, the subjects of complexity and redundancy do not yet arise. By skillful maneuvering, we may also avoid the subject of fatigue.

We assume that the subsystem is new when it is put into operation at the time zero.

Definitions

$F(t)$ = Failure probability, more precisely, the probability that the subsystem will fail within the time interval from zero to t.

$R(t)$ = Reliability, more precisely, the probability that the subsystem will not fail within the time interval from zero to t.

From these definitions follows that

$$F(t) + R(t) = 1 \qquad (2.1)$$

To model these two functions, we concentrate on one of them, say, $F(t)$. It is reasonable to assume that this function has the following properties:

1. $F(0) = 0$

2. $F(\infty) = 1$

3. $\frac{dF}{dt} > 0 \qquad (0 < t < \infty) \qquad (2.2)$

4. $\frac{dF}{dt} = 0 \qquad (t = \infty)$

Conditions 1 and 2 are self-evident. Condition 3 assures monotonicity. Condition 4 mandates asymptotic approach to the maximum value $F(\infty) = 1$ as t goes to infinity.

Conspicuously absent is the condition

$$5. \quad \frac{dF}{dt} = 0 \qquad (t = 0) \tag{2.3}$$

which we nevertheless wish to codify for the purpose of later discussion.

The four conditions (2.2) are satisfied by the following differential equation:

$$dF = (1 - F)\, g(t)\, dt \tag{2.4}$$

This equation makes the differential dF proportional to the differential dt and the factor (1 - F). For generality, the proportionality factor g(t) is still a function of the time. The proportionality of dF to dt is mandated by the calculus of differentiation. The proportionality of dF to (1 - F) is mandated by Conditions 2 and 4.

Condition 3 mandates that

$$g(t) > 0 \qquad (0 < t < \infty) \tag{2.5}$$

Condition 1 will be satisfied by proper selection of the integration constant.

Integration of Equation (2.4) then yields

$$F(t) = 1 - A \operatorname{Exp}\{-G(t)\} \tag{2.6}$$

where, A is the integration constant, and $G(t)$ is defined as follows:

$$G(t) = \int g(t)\,dt \tag{2.7a}$$

or

$$g(t) = \frac{dG}{dt} \tag{2.7b}$$

With Equation (2.6), the original function $g(t)$ is replaced by the function $G(t)$. Since both functions are still largely unspecified, the replacement is purely formal. However, there are two conditions which affect $G(t)$. First, condition (2.5) now reads

$$\frac{dG}{dt} > 0 \qquad (0 < t < \infty) \tag{2.8}$$

Second, Condition 1 has the consequence that

$$0 = 1 - A \operatorname{Exp} \{ - G(0) \} \tag{2.9}$$

Since this equation contains two constants, A and $G(0)$, one of them is freely selectible. We select

$$G(0) = 0 \tag{2.10}$$

This yields $A = 1$. Equation (2.6) then assumes the form

$$F(t) = 1 - \operatorname{Exp} \{ - G(t) \} \tag{2.11}$$

Of course, the reliability now becomes

$$R(t) = \operatorname{Exp} \{ - G(t) \} \tag{2.12}$$

So far, we had smooth sailing, thanks to avoiding complete specification of the function $G(t)$. However, it is obvious that this function is critical to the theory of reliability. It is also to be expected that it bears relevance to the subject of fatigue. While this subject will be addressed in the next section, we now consider the simplest form which $G(t)$ may assume. Obviously, that is the linear form

$$G(t) = a_0 + a_1 t \tag{2.13}$$

From (2.10) then follows that $a_0 = 0$, and from (2.8) follows that $a_1 > 0$. If then we also observe that, for dimensional reasons, a_1 must have the dimension of an inverse time, we may write

$$a_1 = \frac{1}{\tau} > 0 \tag{2.14}$$

Here, τ is a characteristic time of the subsystem which will be interpreted shortly. Equation (2.13) now assumes the form

$$G(t) = \frac{t}{\tau} \tag{2.15}$$

and Equations (2.11) and (2.12) read

$$F(t) = 1 - \mathrm{Exp}\left\{ - \frac{t}{\tau} \right\} \tag{2.16a}$$

$$R(t) = \mathrm{Exp}\left\{ - \frac{t}{\tau} \right\} \tag{2.16b}$$

Before we interpret the nature of the characteristic time τ, we provide one more prerequisite. To that end, we take another look

at the failure probability F(t). Since it is the probability of failure within the finite time interval from zero to t, it is cumulative. This is expressed by the fact that it goes monotonically and asymptotically to one. But if we now form the differential

$$dF(t) = \frac{1}{\tau} \operatorname{Exp}\left\{ -\frac{t}{\tau} \right\} dt \tag{2.17}$$

we see that this is a probability of a different kind. It is the probability of failure within the time interval dt <u>surrounding</u> the time point t. And since dt is infinitesimal, we may as well say that dF(t) is the probability of failure <u>at</u> the time t. Naturally, this probability is infinitesimal. If now we multiply dF(t) by t and then integrate over the entire time interval from zero to infinity, we obtain the expected time of failure or, as it is called, the "Mean Time Before Failure", MTBF. Hence we have

$$\text{MTBF} = \int_0^\infty t \, dF(t) \tag{2.18}$$

which reads

$$\text{MTBF} = \frac{1}{\tau} \int_0^\infty e^{-t/\tau} \, t \, dt \tag{2.19}$$

For later purposes, we dwell for a moment on the method of how to solve the integral. To that end, we define the "moments"

$$M_n = \int_0^\infty e^{-t/\tau} \, t^n \, dt \qquad (n = 0, 1, 2, \ldots) \tag{2.20}$$

For n = 0 and n = 1, we have

$$M_0 = \int_0^{\infty} e^{-t/\tau}\, dt \tag{2.21a}$$

$$M_1 = \int_0^{\infty} e^{-t/\tau}\, t\, dt \tag{2.21b}$$

We then see that (2.19) may be written in the form

$$\text{MTBF} = \frac{M_1}{\tau} \tag{2.22}$$

We also note that the integral for M_0 is straightforward and yields

$$M_0 = \tau \tag{2.23}$$

From this follows that

$$\frac{\partial M_0}{\partial \tau} = 1 \tag{2.24}$$

On the other hand, differentiation of Equation (2.21a) w.r.t. τ yields

$$\frac{\partial M_0}{\partial \tau} = \frac{1}{\tau^2} \int_0^{\infty} e^{-t/\tau}\, t dt \tag{2.25}$$

Because of (2.21b) and (2.24), this reads

$$1 = \frac{M_1}{\tau^2} \qquad (2.26)$$

Therefore, Equation (2.22) yields

$$MTBF = \tau \qquad (2.27)$$

This is an operational interpretation of the characteristic time τ: It is the Mean Time Before Failure.

If we set t = 0 in Equation (2.17), we get

$$\frac{dF}{dt} = \frac{1}{\tau} > 0 \quad (t = 0) \qquad (2.28)$$

If we compare this with Equation (2.3), we see that Condition 5 is NOT satisfied. This is one of the reasons why we separated Condition 5 from Conditions 1 through 4 which, indeed, are satisfied.

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3. FATIGUE

In the preceding section, we selected for the function G(t) occuring in Equations (2.11) and (2.12) the simplest form possible given by equation (2.15). For the purpose of further discussion, we write this function now in the more general form

$$G(t) = \left(\frac{t}{\tau_0}\right)^{\beta} \tag{3.1a}$$

The corresponding reliability function assumes then the form

$$R(t) = \mathrm{Exp}\left\{-\left(\frac{t}{\tau_0}\right)^{\beta}\right\} \tag{3.1b}$$

Equation (3.1a) has two features, one formal, and one substantive. The formal feature is the subscript of τ_0. We did this to facilitate the following discussion. The substantive feature is the exponent β. In the preceding section, this exponent assumed the value one. It appears that values smaller than one do not have any practical importance, but values larger than one do. Therefore, we now adopt the following terminology:

If $\beta = 1$, the system is of Type 1

If $\beta > 1$, the system is of Type 2

In the preceding section, we dealt exclusively with systems of Type 1. However, in the literature, notably in the experimentally oriented literature, we frequently find β- values larger than one, that is, systems of Type 2. The terminology most frequently used in the literature is as follows:

If $\beta = 1$, Exponential Distribution

If $\beta > 1$, Weibull Distribution

However, we prefer our own terminology because it focuses on the systems rather than the distribution functions. We believe that the introduction of the exponent $\beta > 1$ is motivated by experience and observation. The following purely theoretical considerations support this belief.

We start with the question: What is essentially the difference between systems of Type 1 and Type 2? In preparation of the answer, we write Equation (3.1) in the equivalent form

$$G(t) = \frac{t}{\tau(t)} \tag{3.2a}$$

with

$$\tau(t) = \frac{\tau_0^{\beta}}{t^{(\beta - 1)}} \tag{3.2b}$$

Here we replaced the constant MTBF τ_0 by the "time-dependent MTBF" $\tau(t)$. Since Equations (3.2a) and (3.2b) are completely equivalent to Equation (3.1a), the replacement is formal, not substantive. Nevertheless, we shall see that the replacement has a far-reaching heuristic effect.

The time-dependent MTBF $\tau(t)$ is shown in Figure 3-1 for $\beta > 1$. As can be seen from this figure as well as from Equation (3.2b), $\tau(t)$ becomes infinite for $t = 0$. This means that a brand new system has momentarily an infinite MTBF. And this, in turn, has the consequence that dF/dt and dR/dt are zero for $t = 0$. This is expressed

in a deliberately exaggerated manner in Figures 3-2 and 3-3 by the horizontal tangents to the curves $F(t)$ and $R(t)$ at $t = 0$.

That the horizontal tangents in Figures 3-2 and 3-3 are exaggerations follows from the fact that, for

$$1 < \beta < 2$$

that is, $(\beta - 2) < 0$, the second derivatives of $F(t)$ and $R(t)$ go to infinity as t goes to zero. In other words, the zeros of the first derivatives are immediately rescinded by the infinities of the second derivatives.

We now note that systems of Type 2 satisfy Condition (2.3) (Condition 5), whereas systems of Type 1 do not.

As the time increases from zero, the time dependent MTBF $\tau(t)$ begins a monotonic descent from ∞ to zero which it reaches asymptotically as t approaches infinity.

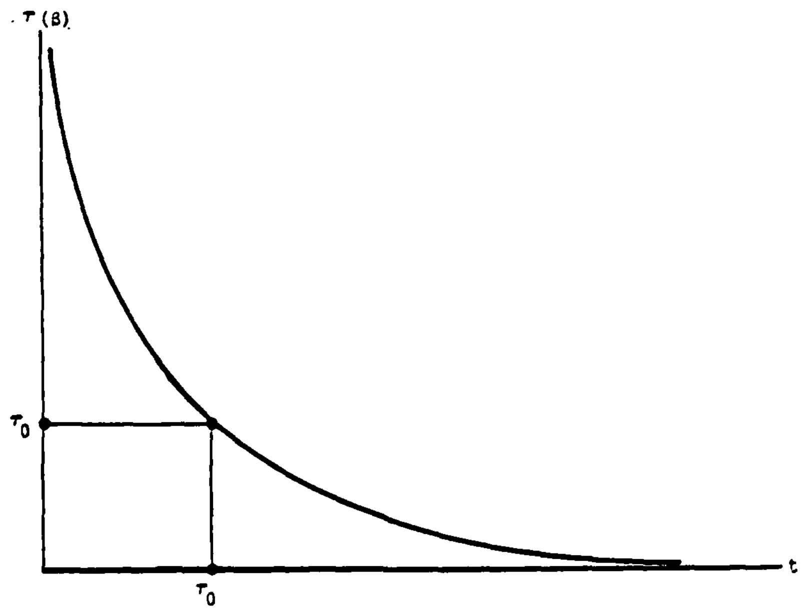

FIGURE 3-1: Time-Dependent MTBF

As can be seen from Figure 3-1, the constant τ_0 achieves a trichotomy of the time scale in the following way:

$$
\begin{array}{lll}
t < \tau_0 & \tau(t) > \tau_0 & \\
t = \tau_0 & \tau(t) = \tau_0 & \qquad (3.3) \\
t > \tau_0 & \tau(t) < \tau_0 &
\end{array}
$$

For a more elaborate discussion of this subject, let us say that the system is "new" if $t < \tau_0$, and that it is "old" if $t > \tau_0$. If we then compare two systems, one of Type 1 and one of Type 2, we arrive at the following conclusion: If the systems are new, the Type 1 system has the smaller MTBF and the larger failure probability; but if the systems are old, the Type 2 system has the smaller MTBF and the larger failure probability. This finding may be explained as follows: Type 2 systems show the signs of aging by having a time-dependent MTBF. In contrast, Type 1 systems have a constant average MTBF, thus hiding the signs of aging. Naturally, when the systems are new, aging is not yet prevalent, thus making the Type 2 system less prone to failure; however, when the systems are old, the signs of aging become prevalent, thus making Type 2 systems more prone to failure. These findings are illustrated in Figures 3-2 and 3-3. They will be corroborated by the considerations that follow.

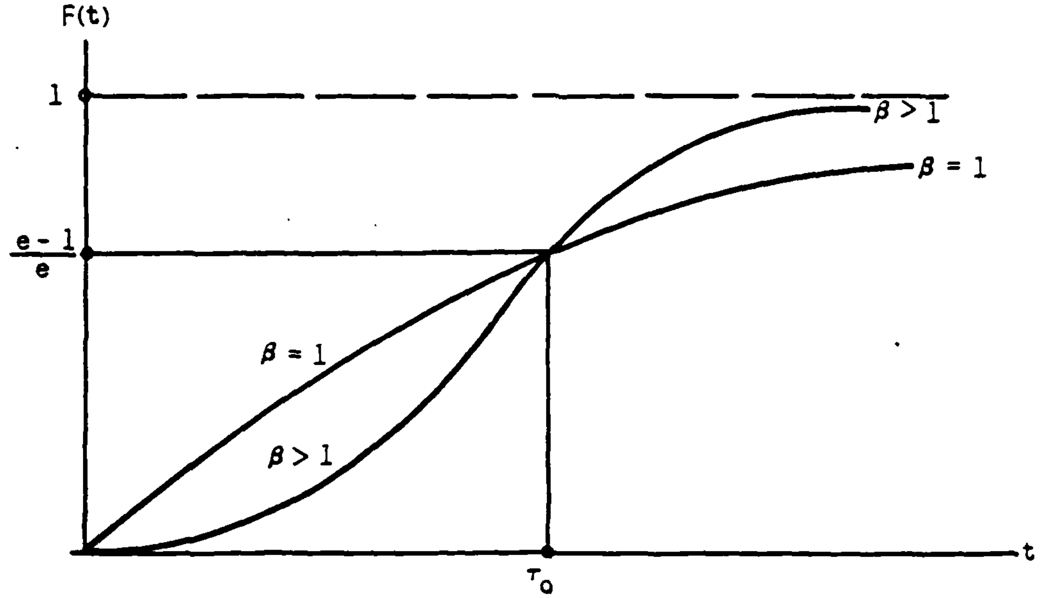

FIGURE 3-2: Failure Probability for Type 1 and Type 2 Systems (Exaggeration)

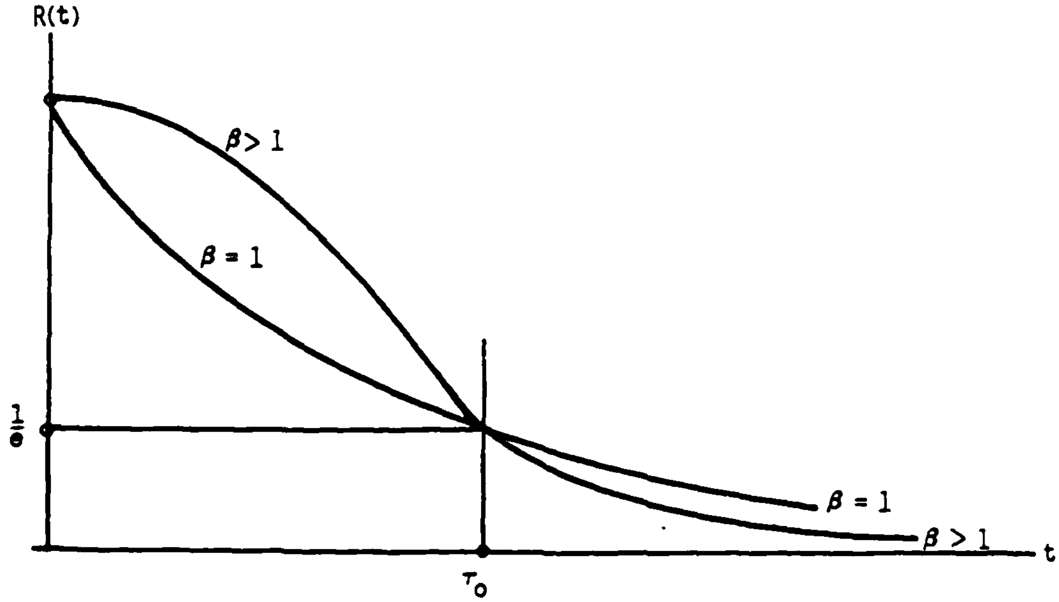

FIGURE 3-3. Reliability for Type 1 and Type 2 Systems (Exaggeration)

We now take a closer look at the way in which systems of Type 1 and Type 2 behave when they have been "aged" by a period of previous operation. To this end, we divide the time continuum according to the proverbial trichotomy of past, present, and future:

$$
\begin{aligned}
t &< t_1 \quad \text{Past} \\
t &= t_1 \quad \text{Present} \qquad (3.4) \\
t &> t_1 \quad \text{Future}
\end{aligned}
$$

We then compare the <u>same system</u> in two <u>different states</u>, or situations.

Situation 1: The Nonaged System

In this situation, the system is brand new at the present time t_1 when it is put in operation. Wanted is the reliability at the future time $t > t_1$.

Here it is important to note that the system has no "history" of previous operation. Therefore, the reliability curve as given by Equation (3.1b) is unaffected, except that it is now shifted the distance t_1 from left to right. If we denote this reliability by $R(t - t_1)$, we have

$$R(t - t_1) = \text{Exp}\left\{ - \frac{(t - t_1)^{\beta}}{\tau_0{}^{\beta}} \right\} \qquad (t > t_1) \qquad (3.5)$$

This curve has exactly the same shape as (3.1b). This is illustrated in Figure 3-4A ($\beta = 1$) and 3-4B ($\beta > 1$) where $R(t - t_1)$ has at $t_1 + \Delta t$ the same value as $R(t)$ at Δt.

To prepare situation 2, we introduce the time

$$t_0 < t_1 \tag{3.5}$$

In other words, t_0 lies in the past.

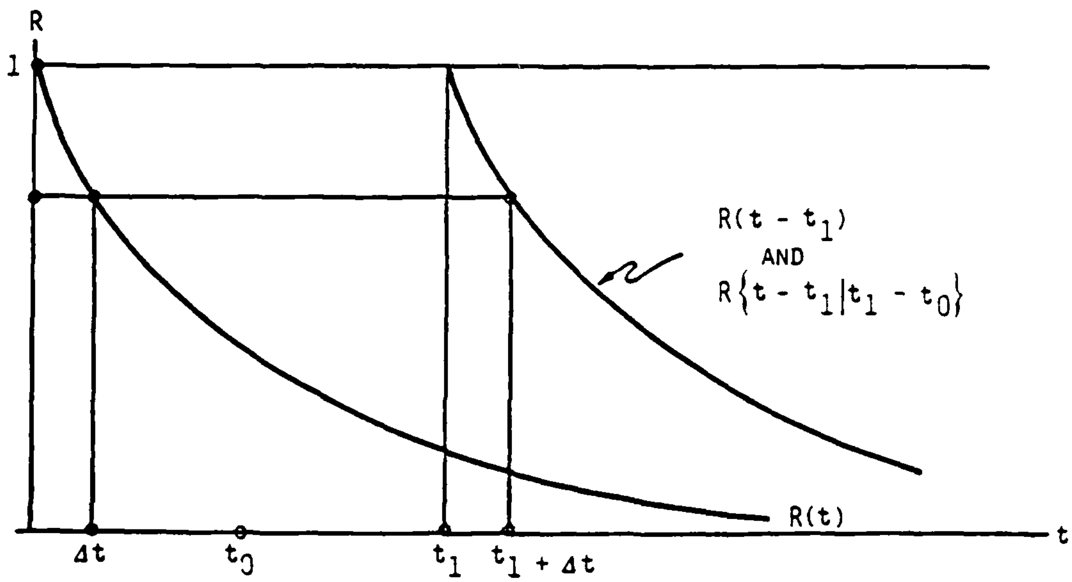

FIGURE 3-4A: No Signs of Aging for $\beta = 1$

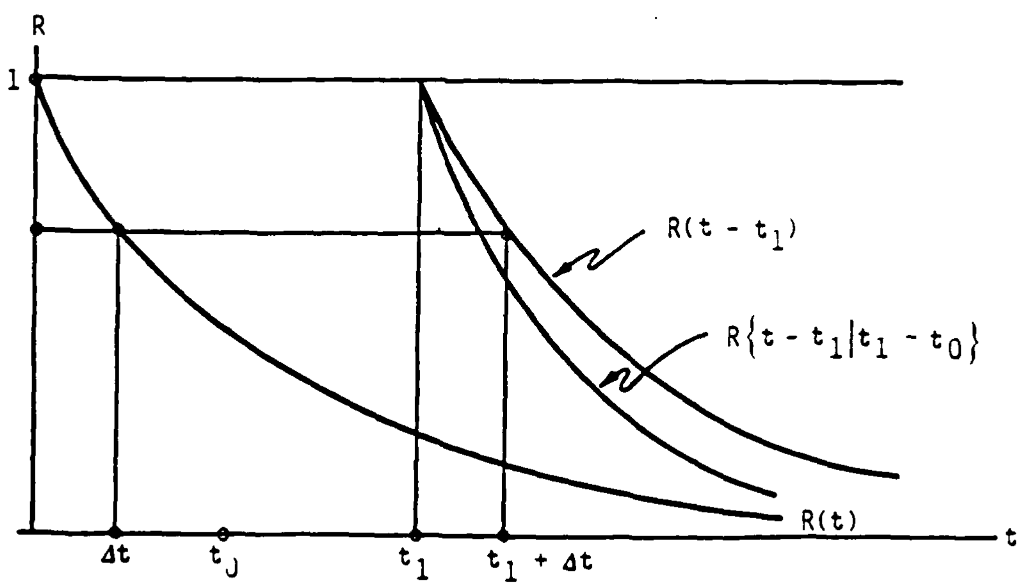

FIGURE 3-4B: Signs of Aging for $\beta > 1$

Situation 2: The Aged System

In this situation, the system was brand new at the past time t_0, at which time it was put in operation. It then operated free of failure until the present time t_1. Wanted is the reliability at the future time $t > t_1$. We denote this reliability by

$R\{t - t_1 \mid t_1 - t_0\}$ which we define as follows:

$R\{t - t_1 \mid t_1 - t_0\}$ = probability that the system operates free of failure from t_1 to t $(t > t_1)$, given that it did operate free of failure from t_0 to t_1 $(t_0 < t_1)$.

To derive this probability, we introduce the joint probability

$R\{(t_1 - t_0) \cap (t - t_0)\}$ which we define as follows:

$R\{(t_1 - t_0) \cap (t - t_0)\}$ = Joint probability that the system operates free of failure from t_0 to t_1 and from t_0 to t $(t_0 < t_1 < t)$.

Here we note that, because of the provision $t_0 < t_1 < t$, the proposition of failure free operation from t_0 to t <u>implies</u> the proposition of failure free operation from t_0 to t_1. Therefore, in the symbol $R\{(t_1 - t_0) \cap (t - t_0)\}$ the term $(t_1 - t_0)$ is redundant, and we have

$$R\{(t_1 - t_0) \cap (t - t_0)\} = R(t - t_0) \tag{3.7}$$

On the other hand, the product rule of the calculus of probability decrees that

$$R\{(t_1 - t_0) \cap (t - t_0)\} = R\{t - t_0 \mid t_1 - t_0\} R(t_1 - t_0) \qquad (3.8)$$

If then we combine Equations (3.7) and (3.8), we obtain

$$R\{t - t_0 \mid t_1 - t_0\} = \frac{R(t - t_0)}{R(t_1 - t_0)} \qquad (3.9)$$

Here, we note the following:

Failure free operation from t_0 to t is the same as failure free operation from t_0 to t_1 AND from t_1 to t. Therefore, we have

$$R\{t - t_0 \mid t_1 - t_0\} = R\{(t_1 - t_0) \cap (t - t_1) \mid t_1 - t_0)\} \qquad (3.10)$$

And here, at the right side, we note that the proposition of failure-free operation from t_0 to t_1 is redundant because it is already stated in the condition compartment. Hence we also have

$$R\{(t_1 - t_0) \cap (t - t_1) \mid t_1 - t_0\} = R(t - t_1 \mid t_1 - t_0) \qquad (3.11)$$

If then we combine Equations (3.10) and (3.11), we obtain

$$R\{t - t_0 \mid t_1 - t_0\} = R\{t - t_1 \mid t_1 - t_0\} \qquad (3.12)$$

And if we substitute this into Equation (3.9), we obtain

$$R\{t - t_1 \mid t_1 - t_0\} = \frac{R(t - t_0)}{R(t_1 - t_0)} \qquad (3.13)$$

We now make use of Equation (3.1b). According to this equation, we have

$$R\ (t - t_0) = \text{Exp}\left\{- \frac{(t - t_0)^{\beta}}{\tau_0{}^{\beta}}\right\} \tag{3.14a}$$

$$R\ (t_1 - t_0) = \text{Exp}\left\{- \frac{(t_1 - t_0)^{\beta}}{\tau_0{}^{\beta}}\right\} \tag{3.14b}$$

If this is substituted into Equation (3.13), we obtain

$$R\ \{t - t_1 \mid t_1 - t_0\} = \text{Exp}\left\{- \frac{(t - t_0)^{\beta} + (t_1 - t_0)^{\beta}}{\tau_0{}^{\beta}}\right\} \tag{3.15}$$

This is the conditional reliability wanted in conjunction with Situation 2.

Before we go ahead, we remind the reader that the previous derivations are predicted on the assumption

$$t_0 < t_1 < t \tag{3.16}$$

We now compare systems of Type 1 and Type 2. For systems of Type 1, that is, for $\beta = 1$, Equations (3.5) and (3.15) assume the form

$$R\ (t - t_1) = \text{Exp}\left\{- \frac{t - t_1}{\tau_0}\right\} \qquad (\beta = 1) \tag{3.17a}$$

$$R\ \{t - t_1 \mid t_1 - t_0\} = \text{Exp}\left\{- \frac{t - t_1}{\tau_0}\right\} \qquad (\beta = 1) \tag{3.17b}$$

Hence we have a result which may or may not surprise the reader:

$$R\{t - t_1 \mid t_1 - t_0\} = R(t - t_1) \qquad (\beta = 1) \tag{3.18}$$

This result says that, for systems of Type 1, the condition of operation during the previous period from t_0 to t_1 is completely irrelevant. What counts is only that the system was still functioning at the time t_1 when the present operation began. This is illustrated in Figure 3-4A for $\beta = 1$ where the curves representing $R(t - t_1)$ and $R(t - t_1 \mid t_1 - t_0)$ are identical.

Before we turn to systems of Type 2, we provide the following auxiliary relation:

$$(1 - x)^{\beta} < 1 - x^{\beta} \qquad (0 < x < 1;\ \beta > 1) \tag{3.19}$$

This relation is true for all values of x and β subject to the conditions stated in the parentheses; for example, for $x = 1/3$ and $\beta = 3/2$, we have

$$(1 - x)^{\beta} = 0.1925;\ 1 - x^{\beta} = 0.4557 \tag{3.20}$$

We then compare Equations (3.5) and (3.15) for $\beta > 1$. We assert that

$$R(t - t_1) > R\{t - t_1 \mid t_1 - t_0\} \qquad (\beta > 1) \tag{3.21}$$

We shall prove the assertion by showing that it leads to the correct relation (3.19). To that end, we infer in conjunction with Equations (3.5) and (3.15) that the assertion (3.21) implies that

$$-\frac{(t - t_1)^{\beta}}{\tau_0^{\beta}} > -\frac{(t - t_0)^{\beta} - (t_1 - t_0)^{\beta}}{\tau_0^{\beta}} \qquad (\beta > 1) \tag{3.22}$$

This becomes

$$(t - t_1)^{\beta} < (t - t_0)^{\beta} - (t_1 - t_0)^{\beta} \qquad (\beta > 1) \qquad (3.23)$$

Here we observe that

$$(t - t_1) = (t - t_0) - (t_1 - t_0)$$

Relation (3.23) then reads

$$\{(t - t_0) - (t_1 - t_0)\}^{\beta} < (t - t_0)^{\beta} - (t_1 - t_0)^{\beta} \qquad (\beta > 1) \qquad (3.24)$$

If we divide this relation by $(t - t_0)$ and introduce the abbreviation

$$x = \frac{t_1 - t_0}{t - t_0} \qquad (3.25)$$

we arrive at our auxiliary relation (3.19). The condition for x is also satisfied since it follows from relations (3.16) and (3.25) that

$$0 < x < 1 \qquad (3.25)$$

This completes the proof of the asserted relation (3.21).

Relation (3.21) states that the conditional reliability of the aged system is smaller than the unconditional reliability of the nonaged system. This is illustrated in Figure 3.4B for $\beta > 1$ where the curve representing $R\{t - t_1 \mid t_1 - t_0\}$ is lower than the curve $R(t - t_1)$.

We have now arrived at the conclusion that systems of Type 2 show the signs of aging, whereas systems of Type 1 hide them. The mathematical mechanism by which systems of Type 2 show the signs of

aging is the exponent $\beta > 1$ or, what amounts to the same, the time-dependent MTBF $\tau(t)$. The mathematical mechanism by which Type 1 systems hide the signs of aging is the exponent $\beta = 1$ or, what amounts to the same, the constant MTBF τ_0.

However, the reader may still have questions. For example:

a. How can an MTBF be time-dependent if, by definition, it is a time average? See for example, Equation (2.18).

b. If Type 1 systems do not age, what then causes them to fail?

c. Is it possible that the exponent $\beta > 1$ is to be attributed to other determinants, notably complexity and redundancy?

The answer to Question a is as follows: The momentary MTBF (t) is the MTBF which the system would display if it remained in its momentary state. In other words, if we have two values of $\tau(t)$, say, $\tau(t_1)$ and $\tau(t_2)$, then these are the MTBFs of the system in two different states, namely the states which the system assumes at the times t_1 and t_2, respectively.

The second question is deliberately rhetorical. We do not say that systems of Type 1 do not age; we merely say that they hide the signs of aging. This leads to the appearance of "spontaneous" failure, that is, failure without cause. A little example will illustrate this.

In nuclear physics, we have the spontaneous decay of radioactive atoms such as uranium or plutonium. This phenomenon is described "pnenomenologically" by the equation of radioactive decay which has the form

$$N(t) = N_0 \text{ Exp} \left\{ - \frac{t}{\tau_0} \right\}$$

Here, N_0 and $N(t)$ are the numbers of nondecayed atoms at the times zero and t, respectively. The point is that the argument of the exponential has the form

$$G(t) = \frac{t}{\tau_0}$$

with $\beta = 1$. This is typical of events which happen spontaneously, that is, without cause. However, such events are only known on the atomic scale. The systems we are dealing with in this report, whether they are systems of Type 1 or Type 2, are not atomic systems, but "thermodynamic" systems, that is, systems consisting of very large numbers of atoms. Such systems do not fail spontaneously, but because of structural fatigue. These are the reasons why we say that systems of Type 1 give the appearance of spontaneous failure, and that they hide the signs of aging.

The third question is properly answered in subsequent sections. At this time, we content ourselves with the terse remark that Sections 4, 5, and 6 will show that complexity and redundancy are properly described by mathematical means other than an exponent $\beta > 1$ or a time-dependent MTBF.

4. COMPLEXITY

To analyze the effect of complexity, we first exclude redundancy. However, there is no need to exclude Type 2 systems. We shall therefore conduct the present investigation for the general case of time-dependent MTBFs.

We assume that all subsystems have equal and total criticality, that is, the system fails as soon as one or more subsystems fail. The MTBF of the i th subsystem is $\tau_i(t)$. Therefore, the failure probability and the reliability of the i th subsystem have the form

$$F_i(t) = 1 - \mathrm{Exp}\left\{-\frac{t}{\tau_i(t)}\right\} \tag{4.1a}$$

$$R_i(t) = \mathrm{Exp}\left\{-\frac{t}{\tau_i(t)}\right\} \tag{4.1b}$$

The time-dependent MTBF is

$$\tau_i(t) = \frac{\tau_i^{\beta i}}{t^{\beta i-1}} \tag{4.2}$$

Here the reader should notice that this equation allows for different β- values for different subsystems.

We assume that the system consists of N subsystems; and we refer to N as the "complexity index". We also assume that the N subsystems are independent of each other. In this case, the reliability of the total system, R(t), is simply the product of the reliabilities of the N subsystems:

$$R(t) = \prod_{i=1}^{N} R_i(t) \tag{4.3}$$

If here we substitute Equation (4.1b) we obtain

$$R(t) = \mathrm{Exp}\left\{-\frac{t}{\tau(t)}\right\} \tag{4.4}$$

where

$$\frac{1}{\tau(t)} = \sum_{i=1}^{N} \frac{1}{\tau_i(t)} \tag{4.5}$$

The failure probability becomes

$$F(t) = 1 - \mathrm{Exp}\left\{-\frac{t}{\tau(t)}\right\} \tag{4.6}$$

Equation (4.5) gives the MTBF of the total system, $\tau(t)$, as a function of the MTBF of the subsystems, $\tau_i(t)$. The following points are worth nothing.

a. The MTBF $\tau(t)$ is time-dependent because the MTBFs $\tau_i(t)$ are time-dependent. If the MTBFs of all subsystems are independent of time, then Equation (4.5) is reduced to

$$\frac{1}{\tau_0} = \sum_{i=1}^{N} \frac{1}{\tau_i} \tag{4.7}$$

In other words, complexity in itself cannot generate time-dependence of the MTBF. Therefore, the exponent $\beta > 1$ found in the literature for complex systems cannot be attributed to the complexity of these systems.

b. Equations (4.5) and (4.7) display the characteristic feature that the <u>inverse</u> of the system MTBF is the sum of the <u>inverses</u> of the subsystem MTBFs. This feature has the following consequence:

Suppose, one subsystem, say, the i th subsystem, becomes entirely unreliable, that is, $\tau_i = 0$. Then $1/\tau_i$ becomes infinite. Then, according to Equations (4.5) and (4.7), $1/\tau(t)$ or $1/\tau_0$ becomes infinite. Then, $\tau(t)$ or τ_0 become zero, that is, the total system becomes entirely unreliable. In other words, the system is only as reliable as its least reliable subsystem, as a chain is only as strong as its weakest link. For more details on this subject, the reader should consult Reference 1.

c. Every MTBF, by definition, is nonnegative. It then follows from Equations (4.5) and (4.7) that, with increasing complexity, that is, with increasing N, the MTBF of the total system decreases. In other words, the higher the complexity of the system, the lower its reliability, and the higher its failure probability. Figure 4-1 illustrates this fact for the failure probability F(t) and various values of the system complexity index N.

d. If the subsystems have different criticality w.r.t. the system, Equations (4.5) and (4.7) are to be modified as follows:

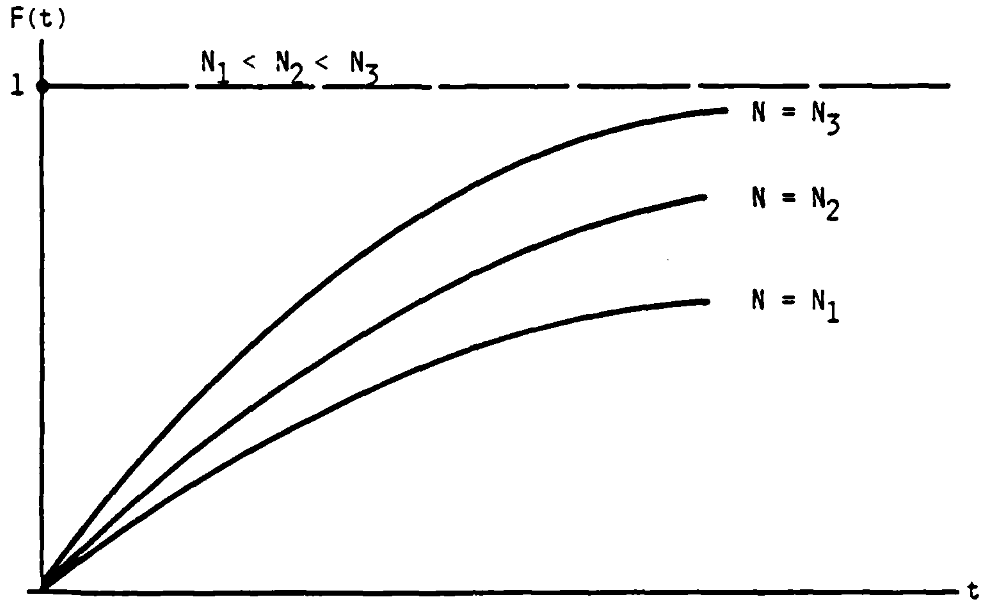

FIGURE 4-1: Increasing Failure Probability with Increasing Complexity

$$\frac{1}{\tau(t)} = \sum_{i=1}^{N} \frac{C_i}{\tau_i(t)} \qquad (4.8a)$$

$$\frac{1}{\tau} = \sum_{i=1}^{N} \frac{C_i}{\tau_i} \qquad (4.8b)$$

Here the C_i are the "measures of criticality" which are subject to the following condition:

$$0 \leq C_i \leq 1 \qquad (4.9)$$

The measures of criticality act as "weights", that is, the larger C_i, the higher is the criticality of the i th subsystem. The extreme cases are:

$C_i = 0$: i th subsystem completely dispensable

$C_i = 1$: i th subsystem totally indispensable

If all subsystems are totally indispensable, we have

$$C_i = 1 \qquad (i = 1, 2, \ldots, N) \tag{4.10}$$

In many practical applications, C_i is the fraction of the time during which proper functioning of the i th subsystem is required for proper functioning of the total system. In these cases, the C_i are "system oriented". The C_i may also be "mission oriented" or "system and mission oriented". For more details, the reader should consult Reference 1.

5. REDUNDANCY

Now turning to redundancy, we exclude complexity in the present section, but we shall combine complexity and redundancy in Section 6. However, it will now be necessary to exclude Type 2 systems. If we did not exclude these systems, then certain integrals could no longer be solved in closed form. This would render the present section so opaque that more would be lost than gained. However, this does not render the present section useless in situations where it is desirable to combine Type 2 systems with redundancy, since the mathematical methods developed in this section may still serve as a guide in those situations.

The redundancy r may assume nonnegative integer values including zero. If a subsystem has redundancy r, then there are r spares, that is, $r + 1$ subsystems of the same kind.

To analyze the effect of subsystem redundancy on the total system, we make one important operational assumption:

The first subsystem spare starts operations when the original subsystem fails. The second spare starts operations when the first fails, and so forth. This assumption excludes external influences such as a hostile environment where all subsystems may be killed at the same time.

The analysts problem may then be formulated as follows:

Given a "multitude" of $r + 1$ equal subsystems (redundancy r), calculate the probability that the <u>multitude</u> will fail within the time t.

For $r = 0$, we already know the answer:

$$F(t/0) = 1 - \mathrm{Exp}\left\{- \frac{t}{\tau}\right\} \tag{5.1}$$

Here, $F(t/0)$ is the failure probability for a multitude of redundancy zero.

Next we calculate the failure probability for a multitude of redudancy one, $F(t/1)$. We now have two subsystems of the same kind. The spare begins operations when the original fails. For the calculation of $F(t/1)$, we now recall the infinitesimal probability $dF(t)$ from Equation (2.14). With the aid of this probability, we can say the following:

The probability that the original subsystem fails <u>at</u> the time t_1 is

$$dF(t_1) = \frac{1}{\tau}\, \mathrm{Exp}\left\{- \frac{t_1}{\tau}\right\} dt_1 \tag{5.2}$$

If we multiply this with the probability

$$F(t - t_1) = 1 - \mathrm{Exp}\left\{- \frac{t - t_1}{\tau}\right\} \tag{5.3}$$

we obtain the joint probability that the first subsystem will fail <u>at</u> the time t_1, and the second will fail <u>within</u> the time from t_1 to t ($t > t_1$). And if we then integrate over t_1 from zero to t, we obtain the probability that the multitude of two will fail within the time from zero to t:

$$F(t/1) = \frac{1}{\tau}\int_0^t \mathrm{Exp}\left(- \frac{t_1}{\tau}\right)\left\{1 - \mathrm{Exp}\left(- \frac{t - t_1}{\tau}\right)\right\} dt_1 \tag{5.4}$$

This becomes

$$F(t/1) = \frac{1}{\tau}\int_0^t \mathrm{Exp}\left(-\frac{t_1}{\tau}\right) dt_1 - \frac{1}{\tau}\ \mathrm{Exp}\left(-\frac{t}{\tau}\right)\int_0^t dt_1 \qquad (5.5)$$

And this becomes

$$F(t/1) = 1 - \left(1 + \frac{t}{\tau}\right) \mathrm{Exp}\left(-\frac{t}{\tau}\right) \qquad (5.6)$$

This is the failure probability for a multitude of redundancy 1. The discussion will be postponed to the end of this section.

Now turning to redundancy 2, we have three factors:

$$dF(t_1) = \frac{1}{\tau}\ \mathrm{Exp}\left(-\frac{t_1}{\tau}\right) dt_1 \qquad (5.7a)$$

$$dF(t_2 - t_1) = \frac{1}{\tau}\ \mathrm{Exp}\left(-\frac{t_2 - t_1}{\tau}\right) dt_2 \qquad (5.7b)$$

$$F(t - t_2) = 1 - \mathrm{Exp}\left(-\frac{t - t_2}{\tau}\right) \qquad (5.7c)$$

These equations display the assumption that the second subsystem starts operations when the first fails, and the third subsystem starts operations when the second fails.

We now have

$$F(t/2) = \frac{1}{\tau^2} \int_{t_1 = 0}^{t_2} \int_{t_2 = 0}^{t} \mathrm{Exp}\left(- \frac{t_1}{\tau} \right) \mathrm{Exp}\left(- \frac{t_2 - t_1}{\tau} \right)$$

$$\left\{ 1 - \mathrm{Exp}\left(- \frac{t - t_2}{\tau} \right) \right\} dt_1 \, dt_2 \tag{5.8}$$

This becomes

$$F(t/2) = \frac{1}{\tau^2} \int_{t_1 = 0}^{t_2} \int_{t_2 = 0}^{t} \mathrm{Exp}\left(- \frac{t_2}{\tau} \right) dt_1 \, dt_2$$

$$- \frac{1}{\tau^2} \mathrm{Exp}\left(- \frac{t}{\tau} \right) \int_{t_1 = 0}^{t_2} \int_{t_2 = 0}^{t} dt_1 \, dt_2 \tag{5.9}$$

Now performing the integration in t_1, we obtain

$$F(t/2) = \frac{1}{\tau^2} \int_0^t \mathrm{Exp}\left(- \frac{t_2}{\tau} \right) t_2 \, dt_2 - \frac{1}{\tau^2} \mathrm{Exp}\left(- \frac{t}{\tau} \right) \int_0^t t_2 \, dt_2 \tag{5.10}$$

Here, the second integral is straightforward. To prepare the solution of the first integral, we employ a method similar to the one expounded in Section 2. To that end, we define:

$$I_0(t) = \int_0^t \mathrm{Exp}\left(- \frac{t^1}{\tau} \right) dt^1 \tag{5.11a}$$

$$I_1(t) = \int_0^t \mathrm{Exp}\left(-\frac{t^1}{\tau}\right) t^1 \, dt^1 \tag{5.11b}$$

With the aid of definition (5.11b), Equation (5.10) may then be written as follows:

$$F(t/2) = \frac{1}{\tau^2} I_1(t) - \frac{1}{2}\left(\frac{t}{\tau}\right)^2 \mathrm{Exp}\left(-\frac{t}{\tau}\right) \tag{5.12}$$

We now have to calculate $I_1(t)$. To that end, we first observe that $I_0(t)$ is straightforward and yields

$$I_0(t) = \tau \left\{1 - \mathrm{Exp}\left(-\frac{t}{\tau}\right)\right\} \tag{5.13}$$

By differentiation w.r.t. τ, we then get

$$\frac{\partial I_0}{\partial \tau} = 1 - \mathrm{Exp}\left(-\frac{t}{\tau}\right) - \frac{t}{\tau} \mathrm{Exp}\left(-\frac{t}{\tau}\right) \tag{5.14}$$

On the other hand, differentiation of (5.11a) w.r.t. yields:

$$\frac{\partial I_0}{\partial \tau} = \frac{1}{\tau^2} \int_0^t \mathrm{Exp}\left(-\frac{t^1}{\tau}\right) t^1 \, dt^1 \tag{5.15}$$

Because of (5.11b), this reads

$$\frac{\partial I_0}{\partial \tau} = \frac{1}{\tau^2} I_1 (t) \tag{5.16}$$

Combination of Equations (5.14) and (5.16) then yields

$$I_1(t) = \tau^2 \left\{ 1 - \mathrm{Exp}\left(- \frac{t}{\tau} \right) \right\} - \tau t \ \mathrm{Exp}\left(- \frac{t}{\tau} \right) \qquad (5.17)$$

And if this is substituted into Equation (5.12), we obtain

$$F(t/2) = \left\{ 1 - \ 1 + \frac{t}{\tau} + \frac{1}{2!}\left(\frac{t}{\tau} \right)^2 \right\} \mathrm{Exp}\left(- \frac{t}{\tau} \right) \qquad (5.18)$$

This is the failure probability for multitudes of redundancy two.

In this way, we may continue and obtain the general formula for multitudes of redundancy r:

$$F(t/r) = 1 - Q(t/r)\ \mathrm{Exp}\left(- \frac{t}{\tau} \right) \qquad (5.19a)$$

where the function Q(t/r) is defined as follows:

$$Q(t/r) = \sum_{k=0}^{r} \frac{1}{k!}\left(\frac{t}{\tau} \right)^k \qquad (5.19b)$$

The corresponding reliability is

$$R(t/r) = Q(t/r)\ \mathrm{Exp}\left(- \frac{t}{\tau} \right) \qquad (5.19c)$$

We now turn to the discussion of these results. We first observe that redundancey does not lead to the exponent $\beta > 1$, but leads to the characteristic functions O(t/r) which multiply the exponential

$$\mathrm{Exp}\left(-\frac{t}{\tau}\right).$$

And since the functions Q(t/r) are larger than one, it follows in conjunction with Equations (5.19a) and (5.19c) that redundancy reduces the failure probability and increases the reliability. Since according to Equation (5.19b) all terms of Q(t/r) are positive, it also follows that

$$Q(t/r+1) > Q(t/r) \tag{5.20}$$

It then follows in conjunction with Equation (5.18a) that

$$F(t/r+1) < F(t/r) \tag{5.21}$$

This means that the multitude with the higher redundancy has the smaller failure probability. This is illustrated in Figure 5-1.

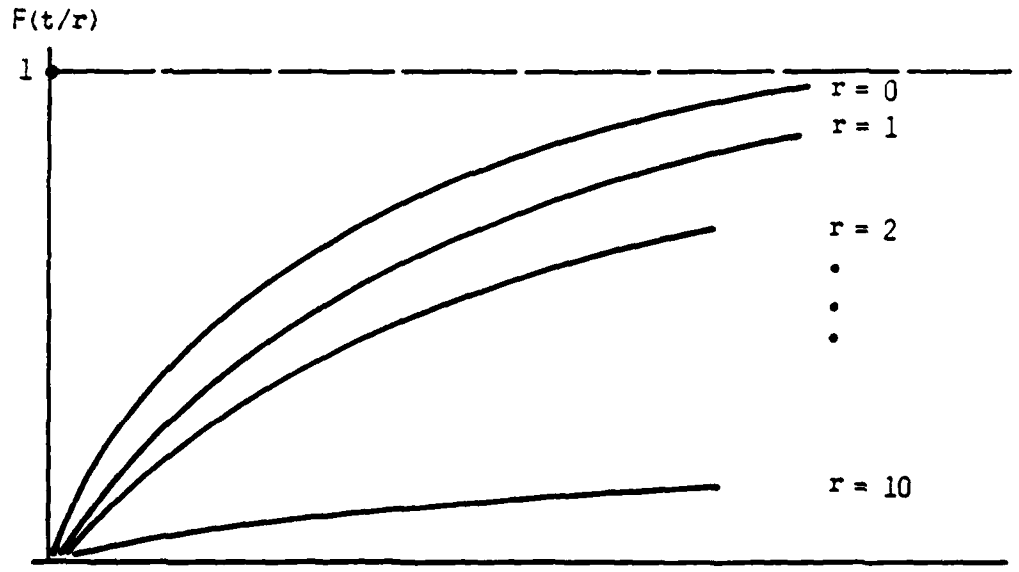

FIGURE 5-1: Failure Probabilities of Multitudes of Various Redundancies

It is interesting to consider the case $r = \infty$, although it does not have much practical importance. We then have

$$Q(t/\infty) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{t}{\tau} \right)^k \qquad (5.22)$$

Here we note that this is the Taylor development of $\mathrm{Exp}\left(+ \frac{t}{\tau} \right)$.

Hence we have

$$F(t/\infty) = 1 - \mathrm{Exp}\left(+ \frac{t}{\tau} \right) \mathrm{Exp}\left(- \frac{t}{\tau} \right)$$

which becomes

$$F(t/\infty) = 0 \qquad (5.23)$$

As one should expect, infinite redundancy reduces the failure probability to zero.

6. COMPLEXITY AND REDUNDANCY COMBINED

As the preceding section, the present section is restricted to systems of Type 1. We consider a system that consists of N subsystems. Each subsystem has its own constant MTBF τ_i, and its own redundancy r_i. Hence for the i th subsystem, the reliability has the form

$$R(t/r_i) = Q(t/r_i)\ \mathrm{Exp}\left\{-\frac{t}{\tau_i}\right\} \qquad (5.1a)$$

with

$$Q(t/r_i) = \sum_{k=0}^{r_i} \frac{1}{k!}\left(\frac{t}{\tau_i}\right)^k \qquad (6.1b)$$

The reliability of the total system is then

$$R(t/r_1, r_2, \ldots, r_N) = Q(t/r_1, r_2, \ldots, r_n)\ \mathrm{Exp}\left\{-\frac{t}{\tau}\right\} \qquad (6.2a)$$

with

$$Q(t/r_1, r_2, \ldots, r_n) = \prod_{i=1}^{N} Q(t/r_i) \qquad (6.2b)$$

and

$$\frac{1}{\tau} = \sum_{i=1}^{N} \frac{1}{\tau_i} \qquad (6.2c)$$

The failure probability is

$$F(t/r_1, r_2 \ldots, r_i) = 1 - R\ (t/r_1, r_2, \ldots, r_i) \qquad (6.3)$$

The reliability and redundancy of subsystems are to a certain extent under the control of the designer. This limited flexibility may be used to trade one against the other. For example, it may be advantageous in terms of weight, volume, or cost, to trade the redundancy of a certain subsystem against higher reliability.

Even more important seems to be the following subject. Since the total system is only as reliable as its least reliable subsystem multitude, it is desirable to make the reliablities of all subsystems multitudes as equal as possible. In so doing, one may either save excess redundancy or gain total system reliability, or both. However, since the redundancy indices r_i can assume only integer values, complete equality of the reliabilities of all subsystem multitudes is rarely possible. Nevertheless, a significant degree of equalization may be achieved by proper selection of the redundancy indices r_i, as the following numerical example will demonstrate.

Suppose the subsystems 1 and 2 have the following MTBFs:

$$\tau_1 = 1.00 \text{ year} \qquad (6.4a)$$

$$\tau_2 = 0.55 \text{ year} \qquad (6.4b)$$

Also suppose that the user of the system is most interested in the period of operation

$$t = 1 \text{ year} \qquad (6.5)$$

If then we select the redundancy indices

$$r_1 = 1 \tag{6.6a}$$

$$r_2 = 2 \tag{6.6b}$$

the corresponding reliabilities of the subsystem multitudes assume the values

$$R_1(t/1) = 0.736 \tag{6.7a}$$

$$R_2(t/2) = 0.726 \tag{6.7b}$$

These equations display the far-reaching equality of reliabilities which we announced. If we selected redundancy indices different from the values (6.6a) and (6.6b), the corresponding reliabilities $R_1(t/r_1)$ and $R_2(t/r_2)$ would be substantially different from each other.

7. RELATED TOPICS

Redundancy is applicable not only to subsystems, but to complete systems as well. For example, a user of satellites may need one satellite in orbit at all times during an extended period of time. For that purpose, he has a certain number of satellites ready to launch; however, to limit launch cost and for other reasons, he does not wish to have more than one satellite in orbit at any given time. Therefore, at the time zero, he launches one satellite. When this satellite fails, he launches the next one, and so forth. Now the question arises: How many satellites does he need to provide single orbiting satellite coverage for the time period from zero to t?

To calculate this number, we need the following probabilities:

$E(t/r)$ = Probability that a multitude of redundancy r will be reliable during the time t, <u>but a multitude of redundancy (r - 1) will fail</u>

We refer to these reliabilities as the "exclusive reliabilities". The exclusion is that part of the definition which we underlined. Whereas the ordinary reliability $R(t/r)$ does not exclude the reliability of a multitude of redundancy $(r - 1)$, the exclusive reliability $E(t/r)$ expressly makes this exclusion.

The exclusive reliabilities $E(t/r)$ and $E(t/r - 1)$ are mutually exclusive since $E(t/r)$ excludes the reliability of a multitude of redundancy $(r - 1)$ whereas $E(t/r - 1)$ assumes reliability of the same multitude. An immediate consequence of this fact is that

$$\sum_{r=0}^{\infty} E(t/r) = 1 \qquad (7.1)$$

This equation which presently follows from purely logical considerations will have to be confirmed later when the functions $E(t/r)$ are known.

With the aid of the exclusive probabilities, we may now calculate the number of satellites needed. To this end, we define:

$\langle r/t \rangle$ = Expected required redundancy to cover the time period t.

Clearly, this is the number wanted, and it is given by the equation

$$\langle r/t \rangle = \sum_{k=o}^{\infty} k\, E(t/k) \qquad (7.2)$$

Since the first term of the sum is zero, we may also write

$$\langle r/t \rangle = \sum_{k=1}^{\infty} k\, E(t/k) \qquad (7.3)$$

We now have to calculate the exclusive reliabilities $E(t/k)$. To that end, we turn to the ordinary reliabilities $R(t/r)$ shown in Figure 7-1 for various values of r.

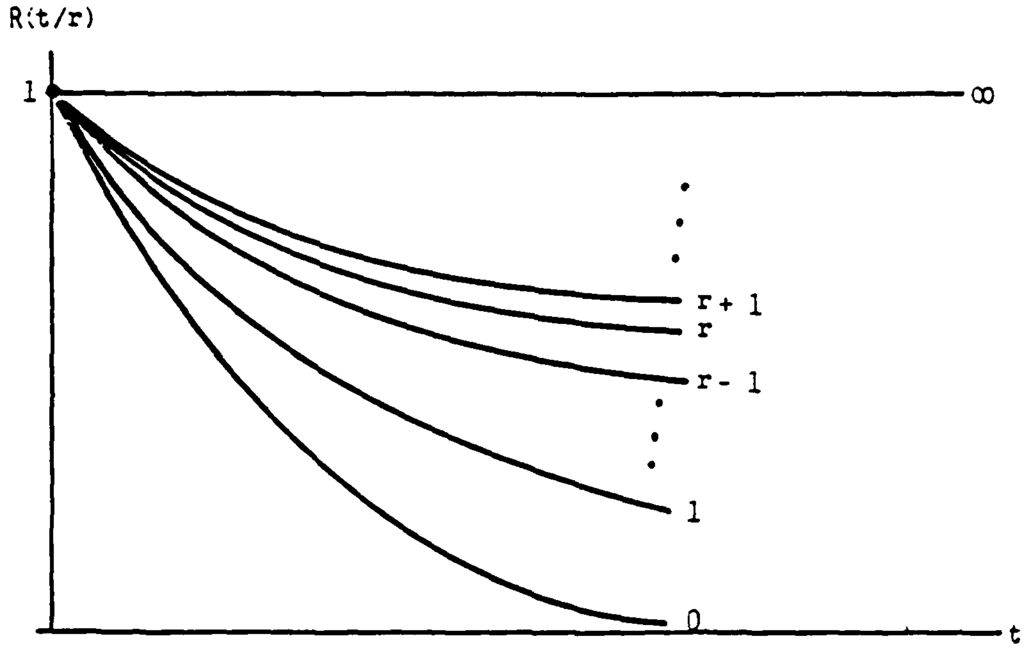

FIGURE 7-1: Ordinary Reliabilities for Various Redundancies

In Figure 7-2, the same reliabilities are represented in the form of a Venn diagram

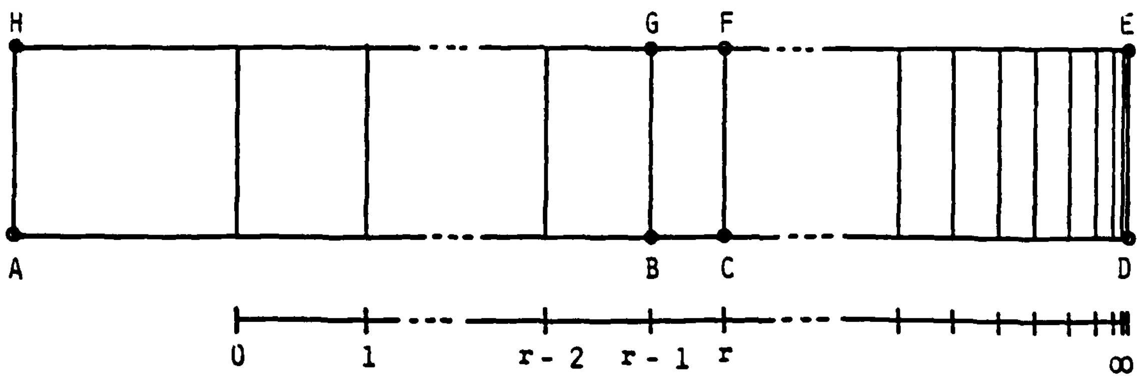

FIGURE 7-2: Venn Diagram of Reliabilities R(t/r)

When compared with the Figure 7-1, the Venn Figure 7-2 has the disadvantage that it does not show the influence of the time, but the advantage that it does show the inclusiveness and exclusiveness of certain probabilities, as will now be discussed.

To explain the Venn diagram, we first recall that $R(t/\infty) = 1$. In view of this fact, it is advantageous (though not compelling) to define the content of the maximal rectangle (ADEH) of Figure 7-2 to be one. We then have

$$R(t/\infty) = \text{Area (ADEH)} = 1 \tag{7.4}$$

All other reliabilities are then represented by areas rather than ratios of areas. We have for example,

$$R(t/r) = \text{Area (ACFH)} \tag{7.5a}$$

$$R(t/r - 1) = \text{Area (ABGH)} \tag{7.5b}$$

But the same areas also represent propositions (or sets). These propositions are:

(ACFH) = A multitude of redundancy r will be reliable within the time interval t

(ABGH) = A multitude of redundancy r-1 will be reliable within the time interval t

From these definitions as well as from the Venn diagram, it is obvious that the proposition (ABGH) implies the proposition (ACFH), that is,

$$(ABGH) \subset (ACFH) \qquad (7.6)$$

In the language of set theory, this relation reads: Set (ABGH) is a proper subset of set (ACFH). The corresponding relation in terms of probabilities is

$$R(t/r - 1) < R(t/r) \qquad (7.7)$$

Next we observe that (ACFH) is the union of (ABGH) and (BCFG), that is,

$$(ACFH) = (ABGH) \cap (BCFG) \qquad (7.8)$$

It can also be seen that the two propositions at the right side of relation (7.8) are mutually exclusive, that is,

$$(ABGH) \cup (BCFG) = \emptyset \qquad (7.9)$$

Here $\emptyset$ is the impossible or contradictory proposition, or the null set.

It now follows from relations (7.8) and (7.9) that

$$\text{Prob } (ACFH) = \text{Prob } (ABGH) + \text{Prob } (BCFG) \qquad (7.10)$$

Here the probability at the left side and the first probability at the right side are already known by virtue of Equations (7.5a) and (7.5b). Hence we only have to interprete the Prob (BCFG). But this is clearly the exclusive reliability E(t/r):

$$E(t/r) = \text{Prob } (BCFH) \qquad (7.11)$$

If now relations (7.5a), (7.5b), and (7.11) are substituted into Equation (7.10), we obtain

$$E(t/r) = R(t/r) - R(t/r - 1) \tag{7.12}$$

Obviously, this relation applies to all values of r except $r = 0$. But from logical arguments or from Figure 7-2, it easily follows that

$$E(t/0) = R(t/0) \tag{7.13}$$

Since the ordinary reliabilities $R(t/r)$ are already known, Equations (7.12) and (7.13) offer an easy way to calculate the exclusive reliabilities $E(t/r)$. But before we carry this out, we form the sum of all $E(t/r)$ from zero to infinity. Equations (7.12) and (7.13) then yield

$$\begin{aligned} \sum_{k=0}^{\infty} E(t/k) = {} & R(t/0) \\ & + \quad R(t/1) - R(t/0) \\ & + \quad R(t/2) - R(t/1) \\ & \quad \vdots \end{aligned}$$

It is not difficult to see that this reduces to

$$\sum_{k=0}^{\infty} E(t/k) = R(t/\infty) = 1$$

This confirms our previous relation (7.1).

To calculate $E(t/r)$, we substitute Equation (5.13c) into Equation (7.12). This yields

$$E(t/r) = \{Q(t/r) - Q(t/r - 1)\}\ \mathrm{Exp}\left\{- \frac{t}{\tau}\right\} \tag{7.14}$$

If here we need Equation (5.18b), we get

$$E(t/r) = \frac{1}{r!}\left(\frac{t}{\tau}\right)^{r}\ \mathrm{Exp}\left\{- \frac{t}{\tau}\right\} \tag{7.15}$$

This equation defines all functions E(t/r) from r = 0 to infinity. If we form once again the sum of all E(t/r), we get

$$\sum_{k=0}^{\infty} E(t/k) = \mathrm{Exp}\left\{- \frac{t}{\tau}\right\} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{t}{\tau}\right)^{k}$$

And this, once again, confirms relation (7.1).

We are now ready to calculate the expected required redundancy <r/t>. Substitution of Equation (7.14) into Equation (7.3) yields

$$\langle r/t\rangle = \mathrm{Exp}\left(- \frac{t}{\tau}\right) \sum_{k=1}^{\infty} \frac{k}{k!}\left(\frac{t}{\tau}\right)^{k} \tag{7.16}$$

This becomes

$$\langle r/t\rangle = \mathrm{Exp}\left(- \frac{t}{\tau}\right) \frac{t}{\tau} \sum_{k=1}^{\infty} \frac{1}{(k - 1)!}\left(\frac{t}{\tau}\right)^{k - 1} \tag{7.17}$$

Here we consider the Taylor development

$$\mathrm{Exp}\left(+ \frac{t}{\tau} \right) = \sum_{m = 0}^{\infty} \frac{1}{m!}\left(\frac{t}{\tau} \right)^{m} \tag{7.18}$$

With the substitution

$$m = k - 1 \tag{7.19}$$

Equation (7.18) assumes the form

$$\mathrm{Exp}\left(\frac{t}{\tau} \right) = \sum_{k = 1}^{\infty} \frac{1}{(k - 1)!}\left(\frac{t}{\tau} \right)^{k - 1} \tag{7.20}$$

If this is substituted into Equation (7.17), we get

$$\langle r/t \rangle = \frac{t}{\tau} \tag{7.21}$$

As a numerical example, we consider

$$t = 10 \text{ years} \tag{7.22a}$$

$$\tau = 1 \text{ year} \tag{7.22b}$$

It then follows that

$$\langle r/10 \rangle = 10 \tag{7.23}$$

Hence the user needs 10 satellites, one for every year.

A slightly different way of addressing the same problem is to calculate the MTBF of a multitude of redundancy r. The definition is

$$\text{MTBF}(r) = \int_0^\infty t \, dF(t/r) \tag{7.23}$$

Here dF(t/r) is the differential of the failure probability F(t/r) defined by Equations (5.18a) and (5.18b). From Equation (5.18a), we obtain

$$dF(t/r) = Q(t/r)\ \text{Exp}\left(-\frac{t}{\tau}\right)\frac{dt}{\tau} - \text{Exp}\left(-\frac{t}{\tau}\right)dQ(t/r) \tag{7.24}$$

To form the differential dQ(t/r), we write Equation (5.18b) in the equivalent form

$$Q(t/r) = 1 + \sum_{m=1}^{r} \frac{1}{m!}\left(\frac{t}{\tau}\right)^m \tag{7.25}$$

The differential then assumes the form

$$dQ(t/r) = \sum_{m=1}^{r} \frac{1}{(m-1)!}\left(\frac{t}{\tau}\right)^{m-1} \frac{dt}{\tau} \tag{7.26}$$

Substitution of Equations (5.18b) and (7.26) into Equation (7.24) then yields

$$dF(t/r) =$$

$$\left\{ \sum_{k=0}^{r} \frac{1}{k!}\left(\frac{t}{\tau}\right)^{k} - \sum_{m=1}^{r} \frac{1}{(m-1)!}\left(\frac{t}{\tau}\right)^{m-1} \right\} \mathrm{Exp}\left(-\frac{t}{\tau}\right)\frac{dt}{\tau} \qquad (7.27)$$

Here, under the first summation sign in the paranthesis, we make the substitution

$$k = m - 1 \qquad (7.28)$$

We then obtain

$$dF(t/r) =$$

$$\left\{ \sum_{m=1}^{r+1} \frac{1}{(m-1)!}\left(\frac{t}{\tau}\right)^{m-1} - \sum_{m=1}^{r} \frac{1}{(m-1)!}\left(\frac{t}{\tau}\right)^{m-1} \right\} \mathrm{Exp}\left(-\frac{t}{\tau}\right)\frac{dt}{\tau} \qquad (7.29)$$

This reduces to

$$dF(t/r) = \frac{1}{r!}\left(\frac{t}{\tau}\right)^{r} \mathrm{Exp}\left(-\frac{t}{\tau}\right)\frac{dt}{\tau} \qquad (7.30)$$

If now Equation (7.30) is substituted into Equation (7.23), we obtain

$$\mathrm{MTBF}(r) = \frac{1}{r!}\,\frac{1}{\tau^{r+1}} \int_{0}^{\infty} \mathrm{Exp}\left(-\frac{t}{\tau}\right) t^{r+1}\, dt \qquad (7.31)$$

Here we recall the moments defined by Equation (2.20). Equation (7.31) then assumes the form

$$MTBF(r) = \frac{M_{r+1}}{r!\ \tau^{r+1}} \tag{7.32}$$

We now see that we need M_{r+1}. To that end, we recall Equation (2.26) which reads

$$M_1 = \tau^2 \tag{7.33}$$

By differentiation of Equations (2.21b) and (7.33) w.r.t. τ, we then get

$$\frac{\partial M_1}{\partial \tau} = \frac{1}{\tau^2} \int_0^{\infty} e^{-t/\tau}\, t^2\, dt \tag{7.34a}$$

$$\frac{\partial M_1}{\partial \tau} = 2\tau \tag{7.34b}$$

It then follows in conjunction with Equation (2.20) for $n = 2$ that

$$M_2 = 2\,\tau^3 \tag{7.35}$$

In this way, we may continue and calculate all higher moments. This leads to the following series:

$$\left.\begin{aligned}
M_0 &= \tau \\
M_1 &= \tau^2 \\
M_2 &= 2\tau^3 \\
M_3 &= 6\tau^4 \\
M_4 &= 24\tau^5 \\
&\circ \quad \circ \\
&\circ \quad \circ \\
&\circ \quad \circ \\
M_n &= n!\ \tau^{n+1}
\end{aligned}\right\} \qquad (7.36)$$

Hence we have

$$M_{n+1} = (n+1)!\ \tau^{n+2} \qquad (7.37)$$

Substitution of (7.37) into (7.32) then yields

$$MTBF(r) = (r+1)\tau \qquad (7.38)$$

Hence we have the series

One Sat: $MTBF\ (0) = \tau$

Two Sats: $MTBF\ (1) = 2\tau$

Three Sats: $MTBF\ (2) = 3\tau$

and so forth. This shows that each satellite increases the MTBF of the multitude by one τ.

8. SUMMARY

1. Complexity and criticality go hand in hand. If a system is complex, and if all subsystems are equally critical, the inverse of the system's MTBF equals the sum of the inverses of the subsystem's MTBFs, as shown by Equations (4.5) and (4.7).

2. If the subsystems have different criticalities, the inverse of each subsystem MTBF is weighted by a measure of criticality C_i, as shown in Equations (4.8a), (4.8b), and (4.9).

3. Redundancy of subsystems is mathematically described by the functions $Q(t/r_i)$ of Equation (6.1b) where r_i is the redundancy of the i th subsystem. These functions multiply the exponentials which constitute the reliabilities, as shown in Equations (6.1a), (6.2a), and (6.2b).

4. The reliability of a subsystem multitude depends on both the subsystem redundancey and the single subsystem reliability. One can trade one against the other without affecting the reliability of the multitude. If the reliabilities of the single subsystems are given, one can equalize the reliabilities of the multitudes by proper selection of redundancies. In this way, one can either save excess subsystem redundancy, or gain total system reliability, or both.

5. Fatigue is the cause of failure. It is realistically described by an exponent $\beta > 1$ in equations such as (3.1a) and (3.2a). A mathematically equivalent description is the time dependent MTBF of Equations (3.2a) and (3.2b). Systems which have these characteristics display the signs of aging by the fact that the reliability from now into the future decreases as the combined duration of past operation increases, as shown in relation (3.21) and Figure 3-4B.

6. We believe that the subjects of complexity, criticality, and redundancy are virtually exhausted; but the subject of fatigue is far from being exhausted. What is needed is a concerted effort to generate more experimental data and to subject them to rigorous mathematical analysis.

END
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